

SLANT SUBMANIFOLDS OF LORENTZIAN ALMOST CONTACT MANIFOLDS

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Abstract

In this paper we study slant submanifolds of Lorentzian almost contact manifolds. We have taken the submanifold as a space like and then defined the slant angle on a submanifold and thus we extended the results of [7] and [8] in this new setting.

1 Introduction

Slant submanifolds were introduced by B.Y. Chen in [5, 6]. These are generalization of both holomorphic and totally real submanifolds of an almost Hermitian manifold. Since then many research articles have appeared on these submanifolds in different known spaces. A. Lotta [8] defined and studied slant submanifolds in contact geometry. Later on, J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez studied slant submanifolds of Sasakian manifolds [4]. Recently, M.A. Khan et.al [7] studied these submanifolds in the setting of Lorentzian paracontact manifolds.

In this paper we defined and studied slant submanifolds of Lorentzian almost contact manifolds. In section 2, we review some formulae for Lorentzian almost contact manifolds and their submanifolds. In section 3, we define slant submanifold assuming that it is space like for Lorentzian almost contact manifolds. we have given in this section characterization theorems for slant submanifold in the setting of Lorentzian almost contact manifolds. The section 4, has been devoted to the study of slant submanifolds of Lorentzian Sasakian manifolds.

2 Preliminaries

Let \bar{M} be a $(2n + 1)$ -dimensional manifold with an almost contact structure and compatible Lorentzian metric, $(\bar{M}, \phi, \xi, \eta, g)$ that is, ϕ is $(1, 1)$ tensor field, ξ is a structure vector field, η is 1-form and g is Lorentzian metric on \bar{M} satisfying [1,3]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0 \quad (2.1)$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \eta(X) = -g(X, \xi) \quad (2.2)$$

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for any $X, Y \in T\bar{M}$, where $T\bar{M}$ denotes the Lie algebra of vector fields on \bar{M} . An almost contact manifold with Lorentzian metric g is called a *Lorentzian almost contact manifold*. From (2.2), it follows that

$$g(\phi X, Y) = -g(X, \phi Y). \quad (2.3)$$

A Lorentzian almost contact manifold is *Lorentzian Sasakian* if [1]

$$(\bar{\nabla}_X \phi)Y = -g(X, Y)\xi - \eta(Y)X. \quad (2.4)$$

It is easy to compute from (2.4) that

$$\bar{\nabla}_X \xi = \phi X. \quad (2.5)$$

Now, let M be a submanifold of \bar{M} , we denote the induced Lorentzian metric on M by the same symbol g . Let $\bar{\nabla}$ and ∇ be Levi-Civita connections on the ambient manifold \bar{M} and the submanifold M , respectively with respect to the Lorentzian metric g then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (2.6)$$

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V \quad (2.7)$$

for any $X, Y \in TM$ and $V \in T^\perp M$, where ∇^\perp is the connection on the normal bundle $T^\perp M$, h is the second fundamental form and A_V is the Weingarten map associated with V as [10]

$$g(A_V X, Y) = g(h(X, Y), V). \quad (2.8)$$

for any $x \in M$, $X \in T_x M$ and $V \in T_x^\perp M$, we write

$$\phi X = TX + NX \quad (2.9)$$

$$\phi V = tV + nV \quad (2.10)$$

where TX (resp. tV) denotes the tangential component of ϕX (resp. ϕV) and NX (resp. nV) denotes the normal component of ϕX (resp. ϕV).

3 Slant submanifolds

Throughout, this section we consider a submanifold M of a Lorentzian manifold \bar{M} such that for all $X \in TM$, $g(X, X) > 0$ or $g(X, X) = 0$ i.e., all the tangent vectors on M are Space like or null like, we shall call these type of submanifolds as *space like* and also we assume that the structure vector field ξ is tangent to the submanifold M . Let M be an immersed submanifold of \bar{M} and for any $x \in M$ and $X \in T_x M$, if the vector field X and ξ are linearly independent then the angle $\theta(X) \in [0, \pi/2]$ between ϕX and $T_x M$ is well defined, if $\theta(X)$ does not depend upon the choice of $x \in M$ and $X \in T_x M$, then M is *slant* in \bar{M} . The constant angle $\theta(X)$ is then

called the *slant angle* of M in \bar{M} and which in short we denote by $\text{Sla}(M)$. The tangent bundle TM at every point $x \in M$ is decomposed as

$$TM = D \oplus \langle \xi \rangle$$

where $\langle \xi \rangle$ is the one dimensional distribution orthogonal to the slant distribution D on M and spanned by the structure vector field ξ .

For any $x \in M$ taking $X \in T_x M$ we put $\phi X = TX + NX$ where $TX \in T_x M$ and $NX \in T_x^\perp M$. Thus defining an endomorphism $T : T_x M \rightarrow T_x M$, whose square T^2 will be denoted by Q . Then tensor fields on M of the type $(1, 1)$ determined by their endomorphisms shall be denoted by same letters T and Q . It is easy to show that for every $x \in M$, $g(TX, Y) = -g(X, TY)$, which implies that Q is anti-symmetric. Moreover, in the following steps we can prove that the eigen value of Q always belong to $[-1, 0]$. For any $X \in T_x M - \langle \xi \rangle$, we get

$$g(QX, X) = -\|TX\|^2$$

but,

$$\begin{aligned} \|TX\| &\leq \|QX\| \\ \|TX\| &\leq \mu \|X\|, \quad \text{and } \mu \in [0, 1]. \end{aligned}$$

Thus we obtain

$$g(QX, X) = -\mu(X) \|X\|^2.$$

That is,

$$g(QX, X) = \lambda(X) \|X\|^2$$

where $-1 \leq \lambda(X) \leq 0$ and λ depends on X . In other words, each eigen value of Q lies in $[-1, 0]$ and each eigen value has even multiplicity.

Now, we have the following theorem.

Theorem 3.1. *Let $x \in M$ and $X \in T_x M$ be an eigenvector of Q with eigenvalue $\lambda(X)$. Suppose X is linearly independent from ξ_x , then,*

$$\cos \theta(X) = \sqrt{-\lambda(X)} \frac{\|X\|}{\|\phi X\|}. \quad (3.1)$$

Proof. For any $X \in TM$ we have

$$\|TX\|^2 = g(TX, TX) = -\lambda(X) \|X\|^2. \quad (3.2)$$

On the other hand by definition of $\theta(X)$ we have

$$\begin{aligned} \cos \theta(X) &= \frac{g(\phi X, TX)}{\|TX\| \|\phi X\|} \\ &= \frac{g(TX, TX)}{\|TX\| \|\phi X\|} = -\lambda(X) \frac{\|X\|^2}{\|TX\| \|\phi X\|}. \end{aligned}$$

Then from (3.2), we obtain that

$$\cos(\theta)(X) = \sqrt{-\lambda(X)} \frac{\|X\|}{\|\phi X\|}.$$

This completes the proof. \square

The following characterization theorem gives the existence of eigen values of the endomorphism Q .

Theorem 3.2. *Let M be a space like slant submanifold of a Lorentzian almost contact manifold \bar{M} and $\theta = Sla(M) \neq \pi/2$, then Q admits the real number $-\cos^2 \theta$ as the only non-vanishing eigen value, for any $x \in M$. Moreover the related eigen space H satisfies $H \subset D$, where $D = Span(\xi_x)^\perp \subset T_x M$.*

Proof. Let $x \in M$, from equation (3.1) $Ker(Q) \neq T_x M$, otherwise $Sla(M) = \pi/2$ which contradict the assumption. So let λ be an arbitrary non-vanishing eigen value of Q and let H be the corresponding eigen space. Now, we have $dim(D) = 2n$ and $dim(H)$ is even, which shows that $dim(H \cap D) \geq 1$. Let $X \in H \cap D$ is a unit vector, then ϕX is also unit vector then from equation (3.1) we obtain

$$\cos \theta = \sqrt{-\lambda(X)},$$

which proves the first part. Moreover, for any $X \in H$, formula (3.1) yields $\|\phi X\| = \|X\|$ which imply that $g(X, \xi) = 0$, hence $H \subset D$. \square

We have noted that, invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion. In case of invariant submanifold $T = \phi$ and so

$$T^2 = \phi^2 = -I + \eta \otimes \xi.$$

While in case of anti-invariant submanifold, $T^2 = 0$. In fact, we have the following general result which characterize slant immersion.

Theorem 3.3. *Let M be a space like submanifold of a Lorentzian manifold \bar{M} such that $\xi \in TM$. Then, M is slant submanifold if and only if there exist a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \quad (3.3)$$

Furthermore, if θ is slant angle of M , then $\lambda = \cos^2 \theta$.

Proof. Necessary condition is obvious, we have to prove the sufficient condition, suppose that there exist a constant λ such that $T^2 = \lambda(-I + \eta \otimes \xi)$, then for any $X \in TM - \langle \xi \rangle$, we have

$$\cos \theta(X) = \frac{g(\phi X, TX)}{\|TX\| \|\phi X\|}$$

$$= -\frac{g(X, T^2X)}{\|TX\|\|\phi X\|} = \lambda \frac{\|\phi X\|}{\|TX\|}. \quad (3.4)$$

On the other hand $\cos \theta(X) = \frac{\|TX\|}{\|\phi X\|}$ and so by using (3.4) we obtain that $\lambda = \cos^2 \theta$. Hence, $\theta(X)$ is a constant angle of M i.e, M is slant. \square

Now, we have the following corollary, which can be easily verified.

Corollary 3.1. *Let M be a space like slant submanifold of a Lorentzian manifold \bar{M} with slant angle θ . Then for any $X, Y \in TM$, we have*

$$g(TX, TY) = \cos^2 \theta (g(X, Y) - \eta(X)\eta(Y)) \quad (3.5)$$

$$g(NX, NY) = \sin^2 \theta (g(X, Y) - \eta(X)\eta(Y)). \quad (3.6)$$

Proof. For any $X, Y \in TM$, then by equation (2.3) we have

$$g(X, TY) = -g(TX, Y).$$

Substituting Y by TY in the above equation we get

$$g(TX, TY) = -g(X, T^2Y).$$

Then by virtue of (3.3), we obtain (3.5). The proof of (3.6) follows from (2.2) and (2.9). \square

4 Slant submanifolds of Lorentzian Sasakian manifolds

In this section we study the slant submanifold of Lorentzian Sasakian manifolds and obtain some interesting results using the equation of curvature tensor.

Theorem 4.1. *Let M be a slant submanifold of a Lorentzian Sasakian manifold \bar{M} . Then Q is parallel if and only if M is anti-invariant.*

Proof. Let θ be slant angle of M in \bar{M} , then for any $X, Y \in TM$ by equation (3.3), we have

$$T^2Y = QY = \cos^2 \theta (-Y + \eta(Y)\xi) \quad (4.1)$$

and

$$Q\nabla_X Y = \cos^2 \theta (-\nabla_X Y + \eta(\nabla_X Y)\xi). \quad (4.2)$$

Taking the covariant derivative of (4.1) with respect to $X \in TM$, we get

$$\nabla_X QY = \cos^2 \theta (-\nabla_X Y + \eta(\nabla_X Y)\xi + g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi). \quad (4.3)$$

Then from equations (4.2) and (4.3) we get

$$(\bar{\nabla}_X Q)Y = \cos^2 \theta (g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi). \quad (4.4)$$

Thus on using (2.5), (2.6) and (2.9) the above equation takes the form

$$(\bar{\nabla}_X Q)Y = \cos^2 \theta (g(Y, TX)\xi + \eta(Y)TX). \quad (4.5)$$

The assertion follows from (4.5). \square

Now, we shall investigate the existence of a slant submanifold using curvature tensor.

Lemma 4.1. *Let M be a submanifold of Lorentzian Sasakian manifold \bar{M} such that ξ is tangent to M . Then for any $X, Y \in TM$, we have*

$$R(X, Y)\xi = (\nabla_Y T)X - (\nabla_X T)Y \quad (4.6)$$

where R is the curvature tensor field associated to the metric induced by \bar{M} on M . Moreover,

$$R(\xi, X)\xi = QX - (\nabla_\xi T)X \quad (4.7)$$

$$R(X, \xi, X, \xi) = g(QX, X). \quad (4.8)$$

Proof. For any $X \in TM$ then from (2.5) and (2.9) we have

$$TX = \nabla_X \xi.$$

Using this fact in the formula $(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y$, we obtain

$$(\nabla_X T)Y = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi.$$

Similarly,

$$(\nabla_Y T)X = \nabla_Y TX - T\nabla_Y X = \nabla_Y \nabla_X \xi - \nabla_{\nabla_Y X} \xi.$$

Substituting these equations in the definition of $R(X, Y)\xi$ it is easy to get (4.6). Rewriting (4.6) for $X = \xi$ and $Y = X$, we obtain

$$R(\xi, X)\xi = (\nabla_X T)\xi - (\nabla_\xi T)X = QX - (\nabla_\xi T)X.$$

Which proves (4.7). Now taking the product with X in (4.7), we get

$$R(\xi, X, \xi, X) = g(QX, X) - g((\nabla_\xi T)X, X). \quad (4.9)$$

The second term of (4.9) will be identically zero as follows

$$\begin{aligned} g((\nabla_\xi T)X, X) &= g(\nabla_\xi TX, X) - g(T\nabla_\xi X, X) \\ &= -g(TX, \nabla_\xi X) + g(\nabla_\xi X, TX) = 0. \end{aligned}$$

Then (4.8) follows from (4.9) using the above fact. \square

Theorem 4.2. *Let M be a submanifold of a Lorentzian Sasakian manifold \bar{M} such that the characteristic vector field ξ is tangent to M . If $\theta \in (0, \pi/2)$ then the following statements are equivalent*

- (i) M is slant with slant angle θ .
- (ii) For any $x \in M$ the sectional curvature of any 2-plane of $T_x M$ containing ξ_x equals $\cos^2 \theta$.

Proof. Assume that the statement (i) is true, then for any $X \perp \xi$ by Theorem 3.3, we have

$$QX = \cos^2 \theta X$$

which by virtue of (4.8) yields

$$R(X, \xi, X, \xi) = \cos^2 \theta. \quad (4.10)$$

Thus (ii) is proved.

Conversely, suppose that (ii) hold then for any $X \in TM$, we may write

$$X = X_\xi + X_\xi^\perp \quad (4.11)$$

where $X_\xi = \eta(X)\xi$ and X_ξ^\perp is the component of X perpendicular to the ξ , using (4.10) and (4.11)

$$\frac{R(X_\xi^\perp, \xi, X_\xi^\perp, \xi)}{|X_\xi|^2} = \cos^2 \theta,$$

or,

$$R(X_\xi^\perp, \xi, X_\xi^\perp, \xi) = \cos^2 \theta |X_\xi|^2. \quad (4.12)$$

Let X be a unit vector field such that $QX = 0$. Then from (4.8) and (4.12)

$$\cos^2 \theta |X_\xi^\perp|^2 = 0. \quad (4.13)$$

If $\cos \theta \neq 0$, then from the above equation $X = X_\xi$. This proves that at each point $x \in M$,

$$Ker(Q) = \langle \xi_x \rangle. \quad (4.14)$$

Moreover, Let A be the matrix of the endomorphism Q at $x \in M$, then for a unit vector field X on M , $QX = AX$, and as $Q(X_\xi) = 0, X = X_\xi$. Then by (4.8) and (4.12)

$$A = \cos^2 \theta I. \quad (4.15)$$

Choosing $\lambda = \cos^2 \theta$, we conclude that for any $x \in M$, This fact together with (4.14) and Theorem 3.3, verifies that M is slant in \bar{M} with slant angle θ . Finally, suppose $\cos \theta = 0$ and X is an arbitrary unit vector field such that $QX = \lambda X$ where $\lambda \in C^\infty(M)$. Then, from (4.8) and (4.12) $g(QX, X) = 0$ that is $\lambda = 0$ and therefore $Q = 0$ which means that M is anti-invariant. \square

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